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20. ABSTRACT

We pursue the analysis of the Hub Operation Scheduling Problem (HOSP) over the finite and infinite horizons. The demand is assumed deterministic and stationary. We deduce the minimum fleet size V_T that satisfies all demands for $1 \leq T \leq \infty$, as well as the optimal schedule that minimizes lost sales for a given fleet size smaller than V_T. Reintroducing the costs of empties and of delayed sales or, equivalently, the cost of empties and the gains from shipments, we resolve the issues of optimal allocation and optimal schedule over a horizon $T \leq \infty$. Finally, we generalize the above results—still under the assumption of deterministic, stationary demands—first to the case in which each city communicates with its two "adjacent" cities (this is the "Wheel" problem) and then to the general network problem in which each terminal may communicate with any other terminal.

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The "Hub" and "Wheel" Scheduling Problems

II. The Hub Operation Scheduling Problem (HOSP): Multi-Period and Infinite Horizon, and the Wheel Operation Scheduling Problem (WOSP)*

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We pursue the analysis of the Hub Operation Scheduling Problem (HOSP) over the finite and infinite horizons. The demand is assumed deterministic and stationary. We deduce the minimum fleet size V_T *that satisfies all demands for* $1 \leq T \leq \infty$*, as well as the optimal schedule that minimizes lost sales for a given fleet size smaller than* V_T*. Reintroducing the costs of empties and of delayed sales or, equivalently, the cost of empties and the gains from shipments, we resolve the issues of optimal allocation and optimal schedule over a horizon* $T \leq \infty$*. Finally, we generalize the above results—still under the assumption of deterministic, stationary demands—first to the case in which each city communicates with its two "adjacent" cities (this is the "Wheel" problem) and then to the general network problem in which each terminal may communicate with any other terminal.*

In Part I of this study[1] we dealt with the "classical" Hub Operation Scheduling Problem (HOSP) as originally formulated by MINAS AND MITTEN.[5] Perhaps the most important feature of that treatment is its

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myopic character, since it considers only the "current" and the "next" scheduling periods.

In Part II we extend the treatment to a horizon of arbitrary length, finite or infinite. We limit the analysis to *deterministic and constant demand* at all terminals, which is in sharp contrast to the probabilistic considerations of Part I. Section 1 treats the case when the objective is to minimize the total number of delayed demand (which is assumed lost), which is equivalent to minimizing the average number of delayed demand. In Section 2 we consider the profit maximization (or cost minimization) problem. Finally, Section 3 is concerned with the generalization of the treatment to the Wheel Operation Scheduling Problem (WOSP).

While Part II can be read independently of Part I, it would be of help if the reader is familiar with Part I, at least for the statement of the HOSP and for insight into the network flow computing algorithm. Of course, the statement can also be gleaned from the paper of Minas and Mitten referenced above, or from ELMAGHRABY.[3]

There are reasonable justifications for considering demand as deterministic and for paying attention to the lost sales case. First, assuming constant demands at all terminals is tantamount to dealing with "certainty equivalents," which are reasonable surrogates for the random variables from a practical point of view. It is often very difficult to obtain the data necessary for determining (with reasonable accuracy) the probability distribution functions of demand. In such cases, certainty equivalents are indeed useful substitutes. Second, the objective of minimizing the average *number* of demand units delayed per period may be equally realistic and applicable as the penalty incurred for such delays because of two reasons: (i) if all customers are equivalent, then the two criteria are, in fact, identical, and (ii) the problem of collecting the cost data with reasonable accuracy may prove insurmountable, so that the penalty criterion becomes operationally infeasible.

In practice, complete equivalence of customers is a rarity and there exists, in all probability, a hierarchy of priorities among them. The above argument then applies to any single priority class.

1. THE MINIMIZATION OF LOST SALES

IN CONSIDERING a horizon longer than two periods (the "current" plus the "next" periods) we encounter several interesting questions, among which we treat only the following two problems:

1. What is the minimum fleet size that satisfies all demand? (Recall that we assume the demand to be deterministic and constant.)

2. Given a fleet size smaller than that specified in 1, what is the optimal scheduling pattern?

Let $N \triangleq \{1, 2, \cdots, n\}$ denote the set of "outlying cities," 0 the hub, $N_0 \triangleq \{0\} \cup N$ the set of all cities, and V the size of the fleet. We shall refer to the set W defined by

$$W \triangleq \{i, j;\ i, j \in N_0 \text{ but } i \text{ and } j \text{ are not in } N \text{ simultaneously}\}. \quad (1.1)$$

Furthermore, we define the constants q_i and r_i as follows:

$$q_i = \max (d_{i0}, d_{0i}) \text{ and } r_i = \min (d_{i0}, d_{0i}). \quad (1.2)$$

Infinite Horizon

Let V_i, $i \in N_0$, be the number of vehicles available at terminal i. The following assertion gives the smallest number of vehicles sufficient to satisfy all the demand without shortage.

ASSERTION 1.1. *It is necessary and sufficient for a minimal fleet of size $V^* = \sum_{i \in N_0} V_i^*$ to satisfy all the demand, that the availabiltiy of vehicles at all terminals be given by*

$$V_i^* = q_i, \qquad i \in N,$$
$$V_0^* = \sum_{i \in N} q_i.$$

Hence $V^* = 2 \sum q_i$. Any fleet of size $V > V^*$ will evidently also satisfy all demand.

The proof of this assertion is by simple contradiction, and is therefore omitted.

COROLLARY 1.1. *The number of empties shipped each period is given by*

$$e_{i0} = V_i^* - d_{i0}, \qquad i \in N,$$
$$e_{0i} = V_i^* - d_{0i}, \qquad i \in N.$$

Thus, in order to incur no delays in the initial period, the minimum number of vehicles available at node i must be $V_i = d_{i0}$, $i \in N$, and $V_0 = V^* - \sum_{i \in N} V_i$. If the proper amount of empties are shipped (under the total availability V^*) to each outlying terminal, delays can be avoided thereafter as well.

In the sequel we shall need the following notation: let M denote the set of outlying cities with more out-shipments than in-shipments per period, i.e.,

$$M \triangleq \{i \in N : d_{i0} \geq d_{0i}\} \text{ and } \overline{M} = N\text{-}M. \quad (1.3)$$

Consider next the question of minimizing the average number of delays (= lost sales) per period for a given number of vehicles $V < V^*$. A schedule

yielding such a minimum will be called optimal, and is given by the following algorithm.

Algorithm 1.1

There are three possible ranges of values of the total fleet size V, with each value demanding a slightly different procedure to obtain the optimal schedule.

1. $V \geq V^*(= 2 \sum_{i \in N} q_i)$. There should be no shortages at all except perhaps in the first two periods due to a mismatch between vehicle availabilities and demands at one or more cities. In period 1, at each city $i \in N$, ship V_i to the hub; at the hub ship q_i to as many cities as possible. In period 2 repeat the same procedure: ship all V_i to the hub, $i \in N$, and ship q_i from the hub to *all* outlying cities. Starting with period 3, ship q_i vehicles from each $i \in N$ to the hub and from the hub to each outlying city. (In this procedure, we assume that the excess fleet $V - V^*$ will be retained at the hub. Any other desired distribution of these excess vehicles can obviously be accommodated.)

2. $2 \sum_{i \in N} q_i > V > 2 \sum_{i \in N} r_i$. Here we must consider two cases:

   (a) $q_i > V_i \geq r_i$, $i \in N$, and $\sum_{i \in N} q_i \geq V_0 > \sum_{i \in N} r_i$. Let $x_{ij}(i, j \in W)$ denote the number of vehicles (full and empty) sent from i to j (see definition (1.1)). For $i \in N$, put $x_{i0} = V_i$ (some of which are empties in case $i \in \overline{M}$; see definition (1.3)). For the hub, put initially $x_{0j} = r_j \; \forall \, j \in N$, and then increase x_{0j} for $j \in M$ up to q_j for as many terminals as possible using the remaining $V_0 - \sum_{i \in N} r_i$ vehicles. Repeat the above allocation each period.

   (b) $q_i > V_i \geq r_i$, $i \in N$, and $2 \sum_{i \in N} q_i > V > 2 \sum_{i \in N} r_i$. This condition encompasses the previous case a as a special case since here V_0 is allowed, at the outset, to be more than $\sum_{i \in N} q_i$ or less than $\sum_{i \in N} r_i$. Hence there are two subcases to consider:

      (i) $V_0 > \sum_{i \in N} q_i$. Put $x_{0i} = q_i$ for $i \in N$; for the outlying terminals use the same schedule as specified in case 2a.

      (ii) $V_0 < \sum_{i \in N} r_i$. In period 1, for $i \in \overline{M}$ put $x_{i0} = \max(r_i, V_i - r_i)$ and put $x_{0i} = \max\{0; r_i - (V_i - r_i)\}$ for as many terminals $i \in \overline{M}$ as possible; for $i \in M$, put $x_{i0} = V_i - (r_i - x_{0i})$, where $x_{0i} = r_i$ for as many terminals as possible (thus, at most one terminal will have $0 < x_{0i} < r_i$ and for the remaining terminals $x_{0i} = 0$). In subsequent periods use the same schedule as specified in case 2a.

3. $V \leq 2 \sum_{i \epsilon N} r_i$. Here again we must distinguish between two cases:

(a) $V_i = r_i$ for $i \epsilon N$ and $V_0 = \sum_{i \epsilon N} r_i$. (Hence the total fleet size is $V = 2 \sum_{i \epsilon N} r_i$.) Schedule r_i shipments between each city $i \epsilon N$ and the hub, and *vice versa*. The number of shipments delayed each period is equal to $\sum_{i \epsilon N}(q_i - r_i) \geq 0$, which is the minimum possible under this fleet size.

(b) $V_i < r_i$ for all $i \epsilon N$ and $V_0 < \sum_{i \epsilon N} r_i$. (Hence the total fleet size is $V < 2 \sum_{i \epsilon N} r_i$.) Schedule V_i shipments between the hub and outlying terminal i each period. The total delays per period is fixed and equal to $\sum_{i,j \epsilon W} d_{ij} - V$.

The above algorithm leads to the following:

ASSERTION 1.2. *There exists an optimal stationary schedule such that the vector of available vehicles in each period is cyclical of length at most two periods.*

For examples of the application of this algorithm (which is intuitively quite transparent), the reader is directed to Reference [1] of Part I.

We now turn to the study of finite horizons.

Finite Horizon

Within the confines of a finite horizon we shall continue to assume that the demands d_{ij}, $i, j \epsilon W$, are constant.

Let T be the number of periods in the horizon, and V^{*T} the minimum fleet size sufficient for no shortages over T.

For $T = 1$, it is obvious that V^{*1} is given by

$$V^{*1} = \sum_{i \epsilon N}(d_{i0} + d_{0i}).$$

For $T = 2$, a little thought reveals that

$$V^{*2} = \sum_{i \epsilon N} q_i + \max\left(\sum_{i \epsilon N} d_{i0}\, ; \sum_{i \epsilon N} d_{0i}\right)$$

where q_i is as defined in Assertion 1.1. Notice that V^{*2} may be $< 2 \sum_{i \epsilon N} q_i$.

For $T \geq 3$, we have

ASSERTION 1.3. $V^{*T} = 2 \sum_{i \epsilon N} q_i, \qquad T = 3, 4, \ldots$

We shall not give a formal proof of this assertion; the interested reader may consult Reference [1] of Part I. However, this assertion should come as no surprise in view of Assertions 1.1 and 1.2. For, if a cycle repeats in, at most, two periods, then it stands to reason that the finite-horizon-no-shortage fleet may differ from the infinite-horizon-no-shortage fleet only if the horizon is of length one or two periods.

In many instances there arises the question of the possibility of "holding empties" at a terminal $i \in N_0$, for one or several periods, in anticipation of future demand. As a consequence of the above assertion, we have

COROLLARY 1.2. *Let y_{it} denote the number of empties "held over" at terminal $i \in N_0$ from period t to period $t + 1$. If $V_T \leq V^{*T}$ then an optimal schedule is s.t. $y_{it} = 0 \; \forall \; i$ and t.*

Consequently, we conclude that in the deterministic constant demand case, since there is no incentive to have a fleet larger than V^{*T} *there shall never be any empties carried over from one period to another at the same terminal.*

Next, we address ourselves to the problem of minimizing the number of shortages over a finite horizon T, given a total fleet size $V < V^{*T}$.

It is assumed that a total fleet of size V can be freely distributed over the terminals initially (i.e., the initial distribution of vehicles is controllable). A total fleet of size V will be categorized into two cases: (1) $0 < V \leq 2 \sum_{i \in N} r_i$, (2) $2 \sum_{i \in N} r_i < V < V^*$. A schedule is called optimal if it minimizes the *total* number of delayed loads (= lost sales) over T. When $T = 1$ or 2, the optimal schedule is easily obtained, and thus the length of the planning horizon will be considered to be 3 periods or more.

For case (1), $0 < V \leq 2 \sum_{i \in N} r_i$, an optimal schedule can be constructed as follows: set $V_i = r_i$, $i \in N$, for as many terminals as possible and assign the remainder of the vehicles (if any) to the hub. Thus, we have $V_0 = V - \sum_{i \in N} V_i \leq \sum r_i$. For the outlying terminals, put $x_{i0} = V_i$ in the first period and return the same number of vehicles to that terminal in the second period. From the hub, send r_i vehicles to outlying terminal i for as many outlying terminals as possible (when $V_0 = \sum r_i$, we can send r_i to *all* outlying terminals from the hub) in the first period and return the same number of vehicles from the outlying terminals to the hub. Repeat this procedure until the end of the planning period T. In this way, all vehicles are kept loaded all the time; thus the schedule is obviously optimal.

For case (2), $2 \sum r_i < V < V^*$, the rationale for constructing an optimal schedule is somewhat more complicated, albeit the final decision rules are extremely simple. In the following we specify these decision rules and omit their detailed justification in the hope that their logic is sufficiently transparent to excuse the omission.

ALGORITHM 1.2. Partition the set M of Equation (1.3) into two subsets as follows:

$$M_1 \triangleq \{i : d_{i0} > d_{0i}\}, \quad M_2 \triangleq \{i : d_{i0} = d_{0i}\}, \quad M_1 + M_2 = M, \qquad (1.4)$$

which leaves $\overline{M} = N - M = \{i : d_{i0} < d_{0i}\}$ as before. Let

$$\delta_i = |d_{i0} - d_{0i}|,$$

the absolute difference in the two-way demands. It is apparent that the pattern of shipments as well as the start and termination cities (we assume that the initial allocation of the fleet is controllable) will both depend on the demands as well as on whether T is even or odd. There are three cases to consider.

(a) M_1 empty; hence $N = M_2 + \overline{M}$.

Initially, assign r_i vehicles to each $i \in N$ and assign the remaining vehicles ($= V - \sum_{i \in N} r_i$) to the hub. In each period, starting with the first, exchange r_i loaded vehicles between the hub and terminal $i \in N$. For a subset of the outlying terminals increase the shipments in the first period to no more than q_i, and return the same vehicles (as empties) to 0 in the following period.

(b) $\overline{M}$ is empty; hence $N = M = M_1 + M_2$.

Initially, assign r_i vehicles to each outlying terminal $i \in N$; and assign $\sum_{i \in N} r_i$ vehicles to the hub. The remaining vehicles ($= V - 2 \cdot \sum_{i \in N} r_i$) are distributed among the outlying terminals in any manner, with each terminal receiving a maximum of q_i vehicles. Let the initial allocation yield V_i vehicles at terminal $i \in N$; $r_i \le V_i \le q_i$. In each period, starting with the first, ship V_i loaded trucks from $i \in N$ to the hub 0, and return the same V_i to the same terminal in the following period (some of which may be empty).

(c) M_1 and $\overline{M}$ are nonempty.

Initially, assign r_i vehicles to each outlying terminal $i \in N$ and assign $\sum_{i \in N} r_i$ vehicles to the hub. Subsequently, there are four subcases to consider:

(i) $\sum_{i \in M_1} \delta_i \ge \sum_{i \in \overline{M}} \delta_i$ and $V \ge 2 \sum_{i \in N} r_i + \sum_{i \in M_1} \delta_i$. Add δ_i to terminal $i \in M_1$ (bringing the total at each terminal to q_i) and add the remaining vehicles ($= V - 2 \sum_{i \in N} r_i - \sum_{i \in M_1} \delta_i$) to the hub.

The q_i vehicles at each terminal $i \in M_1$ travel back and forth between i and 0 (some are empty on the return trip). The vehicles at the hub are apportioned among the terminals as follows: for $i \in M$, assign r_i vehicles to travel back and forth; for $i \in \overline{M}$, assign r_i to all terminals and apportion the remaining vehicles among these terminals in any manner but with no terminal allocated more than q_i vehicles. The total vehicles assigned to

each terminal travel back and forth between these terminals and the hub (some are empty on the return trip).

(ii) $\sum_{i \epsilon M_1} \delta_i \geq \sum_{i \epsilon \bar{M}} \delta_i$ and $V < 2 \sum_{i \epsilon N} r_i + \sum_{i \epsilon M_1} \delta_i$. Add δ_i to as many terminals $i \epsilon M_1$ as possible (bringing their total vehicles to q_i) and assign any remaining vehicles to any terminal $i \epsilon M_1$. Denote the number of vehicles allocated to terminal i by V_i.

For each $i \epsilon M$, ship min (V_i, q_i) back and forth to the hub (some of them are empty on the return trip). For each $i \epsilon \bar{M}$, ship r_i back and forth to the hub. At the hub, ship and return r_i to all outlying cities $i \epsilon N$.

(iii) $\sum_{i \epsilon M_1} \delta_i < \sum_{i \epsilon \bar{M}}$ and $V \geq 2 \sum_{i \epsilon N} r_i + \sum_{i \epsilon \bar{M}} \delta_i$. Add $\sum_{i \epsilon \bar{M}} \delta_i$ vehicles to the hub. Apportion the remaining vehicles ($= V - 2 \cdot \sum_{i \epsilon N} r_i - \sum_{i \epsilon M} \delta_i$) to outlying cities $i \epsilon M_1$ in any manner but with no terminal gaining more than q_i vehicles.

From the hub, ship q_i loaded vehicles to terminals $i \epsilon \bar{M}$ and return the same number of vehicles the next period (some as empties), and ship r_i loaded vehicles to terminals $i \epsilon M$. From terminal $i \epsilon M_1$, ship min (V_i, q_i) loaded vehicles to the hub and return the same number of vehicles the next period (some as empties). From terminals $i \epsilon M_2$ ship r_i loaded vehicles and return them (loaded) the next period.

(iv) $\sum_{i \epsilon M_1} \delta_i < \sum_{i \epsilon \bar{M}} \delta_i$ and $V < 2 \sum_{i \epsilon N} r_i + \sum_{i \epsilon \bar{M}} \delta_i$. Add the difference $(2 \sum_{i \epsilon N} r_i + \sum_{i \epsilon \bar{M}} \delta_i - V)$ to the hub. Apportion this difference among the terminals $i \epsilon \bar{M}$ in any manner, but with no terminal allotted more than q_i vehicles. Let terminal i be allotted V_i vehicles, $i \epsilon \bar{M}$.

From the hub, ship V_i loaded vehicles to terminal $i \epsilon \bar{M}$ and return the same number of vehicles the next period to the hub (some as empties), and ship r_i loaded vehicles to terminals $i \epsilon M$. From terminal $i \epsilon N$ ship r_i loaded vehicles to O and return the same number of vehicles the next period.

In all the cases enumerated above, the procedure is followed in all periods except the last when no empties are shipped except to satisfy an imposed restriction on the terminal location of the vehicles.

We also remark that the subset M_2 does not play any role in the optimization procedure since the allocation, as well as the traffic, is fixed at r_i between the hub and terminal i in all periods.

Example 1.1. Fleet size $V = 36$; $T = 4$, and the demands are as follows:

i	1	2	3	4	5	6	Totals
d_{0i}	4	5	4	2	3	1	19
d_{i0}	2	3	1	6	5	4	21
r_i	2	3	1	2	3	1	12
q_i	4	5	4	6	5	4	28
δ_i	2	2	3	4	2	3	16

Here $M_1 = \{4, 5, 6\}$, $M_2 = \emptyset$, and $\bar{M} = \{1, 2, 3\}$. We also have: $\max(\sum_{M_1} \delta_i, \sum_{\bar{M}} \delta_i) = \max(9, 7) = 9$. Note that

$$2 \sum_i r_i = 24 < V = 36 < 56 = 2 \sum_i q_i.$$

The allocation proceeds as follows:

(i) For the first 24 vehicles $(= 2 \sum_i r_i)$, set $V_i = r_i$, all $i \in N$; and $V_0 = 12(= \sum_i r_i)$.

(ii) Assign the next 9 vehicles to terminals *4*, *5* and *6* which compose the set M_1 so that $V_i \leq q_i$, all $i \in M_1$.

(iii) Assign the remaining three vehicles $(= 36 - 24 - 9)$ to the hub, node *0*.

The shipment of vehicles is always between the hub and the terminal to which they were allocated except in the last period, where shipment is to satisfy demand. The pattern of shipment is shown in Figure 1.

The Fully Utilized Fleet Problem

There remains one more interesting problem deserving investigation: what is the maximum number of vehicles that can be shipped loaded throughout the planning horizon? This question reflects the attitude of a "conservative" shipper, and implies the assumption of no deleterious impact of unsatisfied demand on the demand itself. The answer is simply deduced from the previous results, and is given by

COROLLARY 1.3. *The maximal number of trucks that can be fully loaded at all time is* $2 \sum_{i \in N} r_i$, *assuming constant demand independent of supply of vehicles.*

2. COST MINIMIZATION OR PROFIT MAXIMIZATION

WE RECALL the main distinction between the cost considerations in the myopic case treated in Part I and the cost considerations of interest in the extended horizon case. Here, the cost of shortage of the kth shipment at

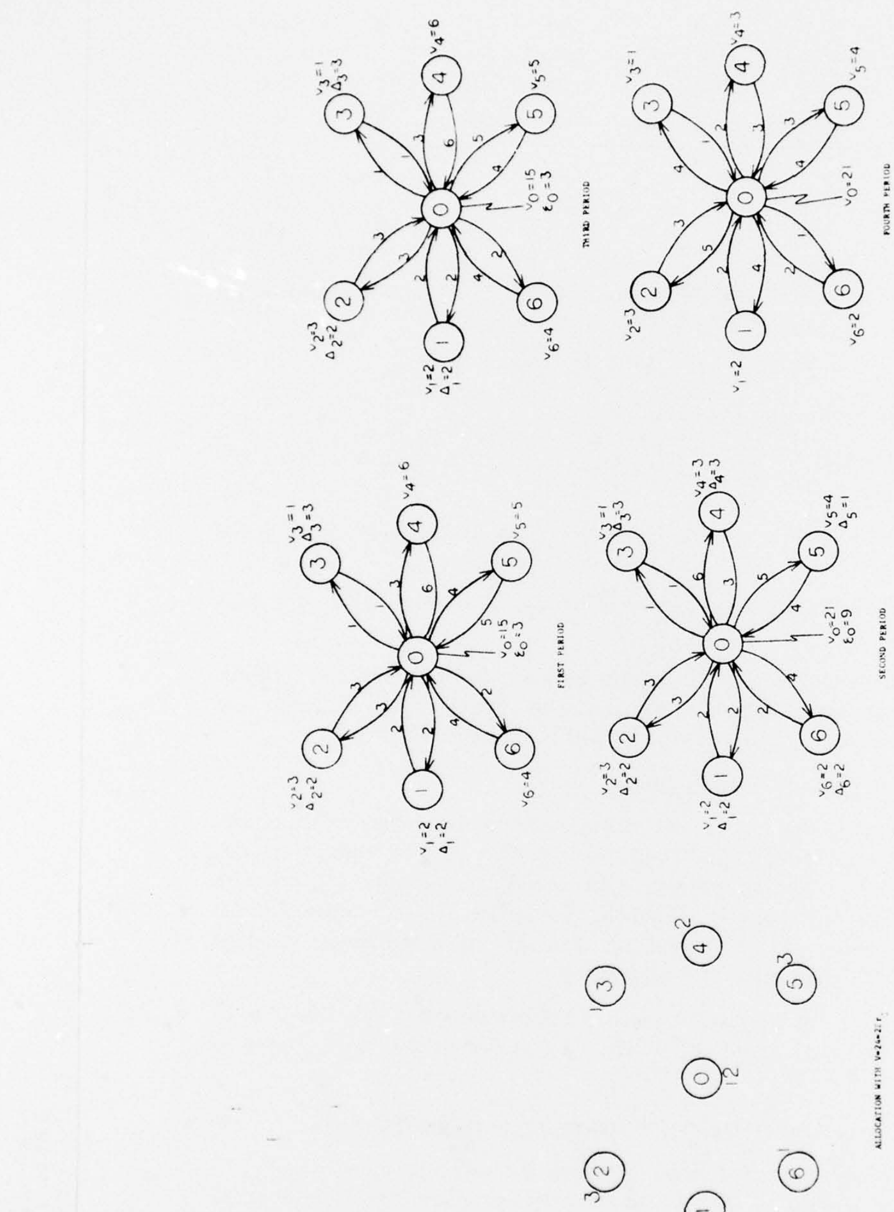


Fig. 1. Solution of Example 1.1: $V = 36$; $T = 4$; total unsatisfied demand = 33 (Δ_i = unsatisfied demand at i; ϵ_i = empties sent from i).

terminal i, denoted by π_{ik}, refers to the cost of *lost sales*. However, the concept of the cost of shipping empties remains the same as in Part I, and we shall continue to denote it by E_i in either direction between terminal i and the hub. These are the only two costs to be considered.

In our subsequent treatment we shall ignore the addition to or subtraction from the fleet of vehicles, previously denoted by A_t. This is for the sake of simplicity of exposition, though the knowledge of additional vehicle availabilities or withdrawals, A_{it}, in period t can be easily incorporated into the optimal allocation and schedule.

Cost minimization can be easily replaced by profit maximization if we replace the negative view of "losing sales" by the positive view of "satisfying demand." Let α_{ik} denote the profit accrued from satisfying the kth demand at terminal $i \in N_0$. Since traffic is bilateral between $i \in N$ and 0, a vehicle is either loaded both ways or is empty one way. Let γ_{ik} denote the maximal net profit in one cycle (which, by assumption, occurs over two periods) when a vehicle is assigned to terminal $i \in N$; then

$$\gamma_{ik} = \begin{cases} \alpha_{ik} + \alpha_{il}^{0} & \text{if vehicle is loaded both ways} \\ \alpha_{ik} - E_i & \text{if vehicle is returned empty.} \end{cases}$$

Here, the subscript k represents the most profitable shipment available at i, and l represents the most profitable shipment available at 0, and the superscript o designates a load available at the hub and destined to terminal $i \in N$. A similar expression can be written for γ_{ik}^{0} at the hub. Clearly, $\gamma_{i1} \geq \gamma_{i2} \geq \cdots$.

In either view, the allocation of vehicles and their scheduling are performed in a sequential fashion, one vehicle at a time. This immediately fixes the subscripts k and l in the above definition of γ_{ik}.

Finite Horizon

In a manner similar to that of Section 2 of Part I, it can be shown that the costs along all arcs of the modified network are convex with respect to the flows; this permits us to formulate the problem as a minimum cost flow problem and to utilize an iterative procedure.

In the description of the procedure we shall opt for a profit maximization objective, in order to demonstrate the utility of the α and γ values introduced above.

It is evident that the total income from a fleet of size $V \leq 2 \Sigma r_i$ and $r_i = \min(d_{i0}, d_{0i})$ is *fixed* since (i) all vehicles are fully loaded in both directions, and (ii) the loads will be chosen to be the most lucrative in the direction in which there is a shortage of vehicles. Therefore, *the discussion of*

optimization commences at a value $V \geq 2 \Sigma r_i + 1$. In which case we must distinguish between two durations of the horizon: T-even and T-odd. Since the optimal schedule for $T = 1$ or $T = 2$ is easily constructed, we shall assume $T \geq 3$.

(1) T (*even*) $= 2\, m;\ m \geq 2$.

Consider the first vehicle (in reality, it is the $(2 \sum_i r_i + 1)$st vehicle). Suppose it is initially available at the hub. There are two possibilities: (i) One can assign it to a load destined to some terminal j from the list of unsatisfied demand at the hub. Evidently this will be the $(r_j + 1)$st load to that terminal, and it must be true that $j \in \overline{M}$. The vehicle must then return to the hub empty if it is to be used for the remainder of the horizon. Consequently, the maximal net profit obtainable from the operation of this vehicle over the horizon will be

$$\max_{j \in \overline{M}}[m(\alpha_{jk}^0 - E_j)], \qquad k = r_j + 1.$$

Alternatively, one may send the vehicle empty to terminal $j \in M_1$ in anticipation of its return loaded; in which case the maximal profit attainable is given by

$$\max_{j \in M_1}[m(\alpha_{jk} - E_j)].$$

Since we seek profit maximization, we search for the terminal maximizing either of these two expressions, i.e., we seek

$$\rho_1 = \max\{\max_{j \in \overline{M}}[m(\alpha_{jk}^0 - E_j)];\ \max_{j \in M_1}[m(\alpha_{jk} - E_j)]\}. \qquad (2.1)$$

Next, suppose that the vehicle is initially available at outlying terminal $j \in M_1$. Such vehicle will also have two possibilities: (i) it can be sent loaded to the hub, and returns empty for subsequent period. This cycle repeats throughout the horizon *except in the last period*, when the vehicle can be utilized to satisfy a demand from the hub to some terminal $p \in \overline{M}$. The maximal realizable profit from such operation is given by

$$\rho_2 = \max_{p \in \overline{M}}(\alpha_{pq}^0) + (m - 1)\,(\alpha_{jk} - E_j) + \alpha_{jk}. \qquad (2.2)$$

Alternatively, the vehicle may be sent empty to the hub in anticipation of its return loaded to the terminal. This cycle repeats throughout the horizon *except in the last two periods*, when the vehicle can be sent loaded to the hub and, from the hub, it is sent loaded to some outlying terminal $p \in \overline{M}$. It is not difficult to see that the maximal realizable profit is also given by ρ_2 of expression (2.2).

We therefore conclude that the optimal allocation of the first vehicle, and its schedule of opertation, is determined by the max (ρ_1, ρ_2), as given by (2.1) and (2.2).

A similar analysis is done for the second available vehicle (in reality, the $(2 \sum_i r_i + 2)$nd vehicle), the 3rd vehicle, and so forth, until V is exhausted.

(2) $T(odd) = 2m + 1, \quad m \geq 1.$

Following similar reasoning we discover that if an additional vehicle is allocated to the hub its maximal profit is

$$\rho_1' = \max\{\max_{i,j\epsilon\bar{M}}[m(\alpha_{ik}^0 - E_i) + \alpha_{jk}^0]; \\ \max_{j\epsilon M_1}[m(\alpha_{jk} - E_j)] + \max_{j\epsilon\bar{M}} \alpha_{jk}^0\}. \quad (2.3)$$

On the other hand, if the vehicle is allocated to an outlying terminal, its maximal profit is

$$\rho_2' = \max\{\max_{j\epsilon N}[m(\alpha_{jk} - E_j) + \alpha_{jk}]; \\ \max_{j\epsilon\bar{M}}[m(\alpha_{jk}^0 - E_j)] + \max_{i\epsilon\bar{M}} \alpha_{ik}\}. \quad (2.4)$$

The maximal allocation is evidently given by max (ρ_1', ρ_2'), which also determines the schedule.

Infinite Horizon

The treatment in Section 2 paves the way to the immediate determination of the optimal schedule in the case of infinite horizon. In particular, if we adopt as an objective the maximization of average profit, then the "corrections" in the ultimate (and penultimate) periods of the horizon in expressions (2.3), (2.3) and (2.4) lose their significance. The optimum is thus seen to reduce to the choice of the terminal $j \epsilon N$ which satisfies the expression:

$$\max\{\max_{j\epsilon\bar{M}}(\alpha_{jk}^0 - E_j);\ \max_{j\epsilon M_1}(\alpha_{jk} - E_j)\}.$$

The optimality of such a stationary policy is a direct consequence of the finiteness of the state space (as represented by the various possible allocations of the fleet of size V over the terminals of the system) and the decision space in each period; see Reference [2].

3. THE WHEEL OPERATION SCHEDULING PROBLEM (WOSP)

THE WOSP is the first generalization of the HOSP treated in Part I and in Sections 1 and 2 of Part II. The assumption of limited bilateral operation between the hub and outlying terminals previously assumed in HOSP is partially relaxed: an outlying terminal is now allowed to dispatch vehicles on hand to two other "adjacent" outlying terminals, where "adjacency" to j is defined as terminals $j - 1$ and $j + 1$; and the numbers are taken around a circle in a "round robin" fashion. Thus terminal 1 is adjacent to terminals

n and 2; terminal 2 is adjacent to terminals 1 and 3, and so forth. Let $\mathcal{C}_j$ denote the set of terminals "communicating" with terminal $j \in N_0$; then

$$\mathcal{C}_0 = N, \text{ since the hub communicates with all } i \in N,$$

$$\mathcal{C}_j = \{0; j-1, j+1\}, \qquad j \neq 1, n,$$

$$\mathcal{C}_1 = \{0; n; 2\}; \ \mathcal{C}_n = \{0; n-1, 1\}.$$

A slight modification in notation from that utilized thus far is needed. The set W defined in definition (1.1) is now expanded to include the set of terminals adjacent to terminal i. We denote the expanded set by $\hat{W}$, i.e.,

$$\hat{W} = \{i, j : i \in N; j \in 0 \cup \mathcal{C}_i\}$$

The Minimization of Lost Sales

Here we concern ourselves with the *smallest fleet that satisfies all demand.*

Consider first a finite horizon of length T. Let a node (ik) represent terminal $i \in N_0$ in period k, $k = 1, 2, \cdots, T$. Introduce the fictitious source s and terminal t. Let $x_{\sigma\eta}$ denote the number of loaded vehicles from node $\sigma \leftrightarrow (ik)$ to node $\eta \leftrightarrow (j, k+1)$. Then the determination of the minimum number of vehicles V_T that satisfy the demands at all times during the horizon of length T is given by the following LP:

$$\min V_T$$

$$\text{s.t. } \textstyle\sum_{j \in \mathcal{C}_i} (x_{((ik),(j,k+1))} - x_{((j,k-1),(ik))}) = 0, \ \forall\, i \in N_0; j \in \mathcal{C}_i, k = 0, 1, \cdots, T-1$$

$$\textstyle\sum_{i \in N_0} x_{(s,(i1))} = V_T = \sum_{i \in N_0} x_{((iT),t)}$$

$$x_{((ik),(j,k+1))} \geq d_{ij}; \ \forall\, i \in N_0, j \in \mathcal{C}_i, \qquad k = 0, 1, \cdots, T-1$$

$$\text{all } x_{(\sigma\eta)} \geq 0 \text{ and integer.}$$

This is a straightforward flow minimization problem subject to lower bounds on the flow on each arc. It is well-known (see, e.g., Reference [4]) that the minimum value V^{*T} is equal to the maximum sum of the lower bounds of all cut-sets between nodes s and t. A labeling procedure yields the desired optimum V^{*T} directly. It initiates from node t since we will start with a large flow through the network and find a sequence of flow-decreasing paths. The labeling procedure is as follows:

Step 0. Generate an initial feasible flow, large enough (see below).
Step 1. Label t with (ι, ∞).
Step 2. Label node (iT), $i \in N_0$, with $(t, x_{((iT),t)})$.
Step 3. (Backward labeling) For each unlabeled node $i \in \mathcal{C}_j$, $h \leftrightarrow (i, k-1)$, $l \leftrightarrow (jk)$, where $x_{hl} > d_{hl}$, assign the label $(l^+, \epsilon(h))$ where $\epsilon(h) = \min [\epsilon(j), x_{hl} - d_{hl}]$.

Step 4. (Forward labeling) For each unlabeled node $j \in \mathcal{C}_i$, $h \leftrightarrow (i, k-1)$, $l \leftrightarrow (jk)$, assign the label $(h^-, \epsilon(l))$ where $\epsilon(l) = \epsilon(h)$.

Step 5. If node $h \leftrightarrow (i1)$, $i \in N_0$, is labeled with $\epsilon(h)$, then label s with $[h, \min\{\epsilon(h), x_{sh}\}]$.

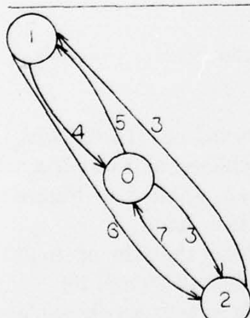
Step 6. If s is not labeled (nonbreakthrough), the current flow is optimal. If s is labeled (breakthrough) decrease the flow along "forward" arcs in the "flow path"[1] by $\epsilon(s)$, and increase the flow along "reverse" arcs by $\epsilon(s)$. Erase all labels and return to Step 2. (Here a "forward" arc refers to an arc whose direction is the same as the path from s to t; a "reverse" arc has its arrow in the opposite direction to the path.)

Any arbitrary large flow can serve as initial feasible flow since there are no upper bounds on the arc capacities, only lower bounds (equal to d_{ij}). However, $V^T = 2\sum_{i,j \in W} q_{ij}$ is sufficient since it can be easily seen that the desired minimum, V^{*T}, can never exceed this number. Indeed, the desired minimum is bound as follows: $2\sum_{i,j \in \hat{W}} r_{ij} \leq V^{*T} \leq 2\sum_{i,j \in \hat{W}} q_{ij}$.

Example 3.1

Let $N_0 = \{0, 1, 2\}$, i.e., there are only two outlying terminals, and $T = 3$. The demand is as shown in Table I. The initial feasible solution is shown in Figure 2(a), in which we introduced flow equal to $2\sum q_{ij} = 36$. In Figure 2(b) the labeling indicates that there are three independent flow paths, with reductions equal to 1 each. When these reductions are effected the result is the flow shown in Figure 2(c). Another labeling step detects

TABLE I



DEMANDS

$i \backslash j$	0	1	2		0	1	2	
0		5	3			5	7	
1	4		6	r_{ij}	4		6	q_{ij}
2	7	3			3	3		
		α_{ij}				$2\sum r_{ij} = 20$ $2\sum q_{ij} = 36$		

[1] This is the complete path between s and t with all its nodes labeled with at least $\epsilon(s)$.

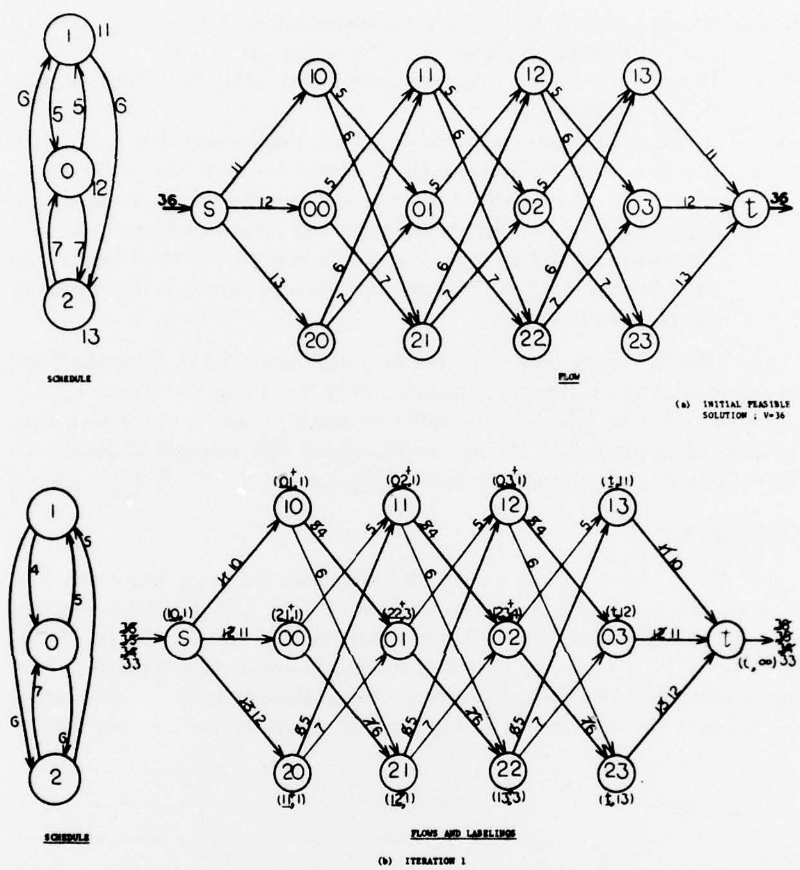
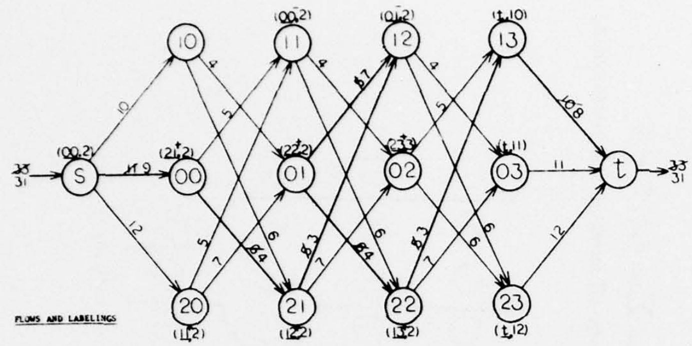


Fig. 2. Labeling procedure for Example 3.1.

the flow path, s, *21*, *12*, *01*, *22*, *13*, t (note the forward arc (12, 01) with $\epsilon(s) = 2$). The total flow is now reduced to only 31 vehicles, and we obtain the arc flows shown in Figure 2(d). The labeling step, applied to Figure 2(d), results in nonbreakthrough; hence the optimal is in hand.

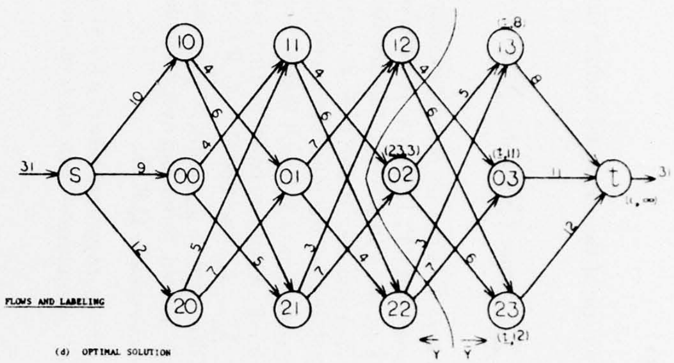
The minimal cutset is also shown on Figure 2(d). Note that the optimal value $y^{*T} = 31$ indeed lies between the specified bounds of 20 and 36. To the extent that it is strictly less than $2 \sum q_{ij}$, the difference is attributable to the balance brought into the system due to the capability to ship along the "rim" of the "wheel."

Although the above labeling algorithm is quite simple and straight-



FLOWS AND LABELINGS

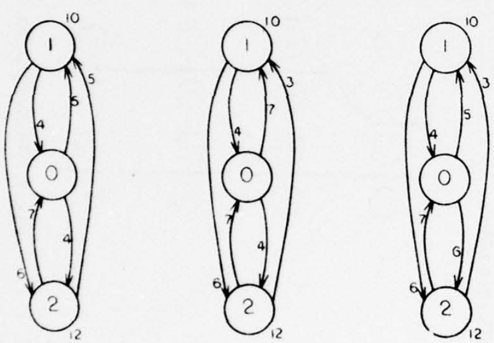
(c) ITERATION 2



FLOWS AND LABELING

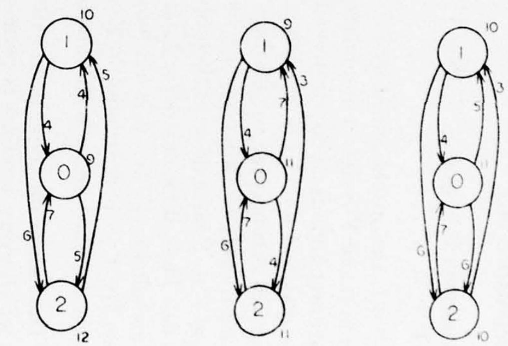
(d) OPTIMAL SOLUTION

SCHEDULE



1st PERIOD 2nd PERIOD 3rd PERIOD

SCHEDULE



1st PERIOD 2nd PERIOD 3rd PERIOD

forward, we present some results on the nature of the cut which reduce the computation of V^{*T}.

ASSERTION 3.1. *If* $n \leq 3$, then $V^{*T} = \sum_{i \epsilon N_0} \max \{\sum_{j \epsilon \mathfrak{C}_1} d_{ij}, \sum_{j \epsilon \mathfrak{C}_1} d_{ji}\}$ $\forall\, T \geq 2$.

Proof. Denote the cutset by $(Y, \bar{Y})$, where $\bar{Y}$ is the set of labeled nodes (from t) and Y is the set of nonlabeled nodes. Let $l(Y, \bar{Y})$ denote the sum of the lower bounds on flow from nodes in Y to nodes in $\bar{Y}$. Then the cut with the maximal $l(Y, \bar{Y})$ is the one sought. For the case where $n \leq 3$, every terminal is connected to all others. As is clear from the labeling procedure, there are only five kinds of cuts, as depicted in Figure 3. Recall that all arcs from Y to $\bar{Y}$ have flows at their lower bounds, which are equal to d_{ij} in our model. Thus, if for any $i \epsilon N_0$, $\sum_{j \neq i} d_{ij} \leq \sum_{j \neq i} d_{ji}$, then node $h \leftrightarrow (i, T-1)$ is included in Y; i.e., it is not labeled. Otherwise, if $\sum_{j \neq i} d_{ij} > \sum_{j \neq i} d_{ji}$, then $h(i, T-1)$ is included in $\bar{Y}$; i.e., it is labeled. Summing over all nodes of the modified network the conclusion is obtained. Q.E.D.

The logic of the above proof extends to the more general case of a network in which every terminal is permitted to communicate with *all* terminals. We thus have

COROLLARY 3.1. (*A Generalization*) *If all terminals communicate with each other, then*

$$V^{*T} = \sum_{i \epsilon N_0} \max \{\sum_{j \neq i} d_{ij}, \sum_{j \neq i} d_{ji}\}, \; T \geq 3.$$

This solves the case of a *general network with all terminals communicating with each other.*

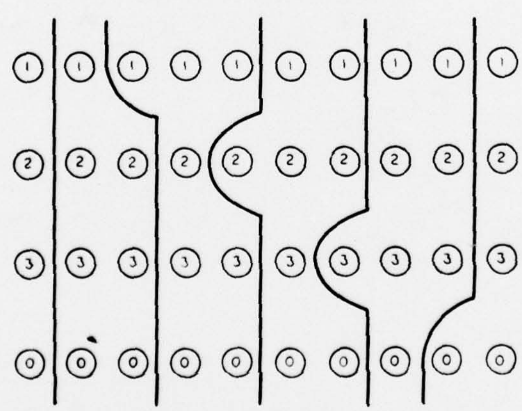


Fig. 3. The five possible cuts with three outlying terminals.

The above assertion specifies the optimum fleet over a finite horizon. In the case of a very long, or unbounded horizon, we have:

ASSERTION 3.2. For all $T \geq 3$, $V^{*T} = \text{constant} = V^{*3}$.

Proof. In searching the cut with the maximal $l(Y, \bar{Y})$, the network for $T = 3$ provides all the possible cuts for the case of $T \geq 3$ since the cut can traverse at most two periods. Thus we search all possible cuts when we search the network with $T = 3$. When $T \geq 3$, because of the stationarity of demands, no cut with larger $l(Y, \bar{Y})$ can be obtained. Q.E.D.

Thus, for any size WOSP, we can find V^{*T} for any T by simply computing for $T = 3$. (Note that V^{*1} and V^{*2} are immediate.)

Finally, there remains the issue of minimizing the lost sales given a fixed fleet V, $2 \sum_{(ij)} r_{ij} < V < V^*$, over the finite or the infinite horizon. It is easily seen that, for the finite horizon, the problem is solved through a cost-minimization model similar to that proposed in Section 1. Furthermore, a cycle will be discerned for $T \geq 3$ which will repeat forever in the case of an infinite planning horizon. Such a cycle can be demonstrated to be at most of period $(n + 1)$ where n is the number of outlying terminals (see Assertion 1.2). Consequently, a cost minimization problem over a *finite* horizon of $T \geq 2(n + 1)$ will guarantee the detection of the optimal allocation and the corresponding optimal schedule.

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